Brief Reports

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$S^1 \times S^2$ wormholes

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We solve the Euclidean Einstein equations with three-index antisymmetric and two-index antisymmetric (electromagnetic) tensors for monopole configurations on a space with three-surfaces of topology $S^1 \times S^2$ and describe wormhole solutions. We show that these wormholes and the one of Giddings and Strominger, Hawking, Halliwell and Laflamme, and Myers can be obtained by slicing five-dimensional spaces with horizons.

I. INTRODUCTION

Giddings and Strominger recently wrote down a solution to the Einstein equations with a third-rank antisymmetric tensor field matter source which represents a wormhole, or bridge, connecting two asymptotically flat regions of Euclidean space.[1] This solution was generalized by Myers to the case of a nonzero positive cosmological constant, in arbitrary dimensions.[2] Hawking[3] and Halliwell and Laflamme[4] investigated the possibility that the matter source be a massless conformally coupled scalar field instead of an antisymmetric tensor field and Dowker[5] has studied the electromagnetic case. These solutions have been useful in making more concrete discussion of the mechanism recently proposed by Coleman for the vanishing of the cosmological constant.[6]

In both cases, the wormholes have topology $R \times S^3$. In this Brief Report we investigate the possibility of having wormholes with baby universes of topology $S^1 \times S^2$. We look for solutions with three- and two-antisymmetric-index tensor fields. We conclude by showing that all the known wormholes are related to five-dimensional spaces with horizons.

II. WORMHOLE SOLUTIONS

We assume the following form of the Euclidean action for fixed initial and final three-geometries:

$$I_E = \int d^4x\sqrt{g}\left[-\frac{R}{16\pi G}+\frac{f^2H^2}{3!}+\frac{e^2F^2}{2!}\right] + \int d^3x\sqrt{h}\,\frac{K}{8\pi G}\;, \tag{2.1}$$

where R is the Ricci scalar, H is an antisymmetric three-index tensor with coupling constant f^2, and finally F is the usual electromagnetic tensor with coupling e^2.

In this Brief Report we investigate homogeneous solutions which extremize the action (2.1) and have a metric of the form

$$ds^2=\sigma^2[N^2(\tau)d\tau^2+a^2(\tau)dr^2+b^2(\tau)d\Omega_2^2]\;, \tag{2.2}$$

where r is the coordinate of a submanifold of topology S^1 and has range $[0,2\pi]$, $d\Omega_2^2$ is the metric on the two-sphere S^2 and $\sigma^2=G/2\pi$. Moreover we will investigate "monopole" solutions for the field H, i.e., H is proportional to the volume element of the surface of topology $S^1\times S^2$ and that F is the usual monopole on the two-sphere:

$$H=\frac{\sigma Q_a}{4\pi f}\sin\theta\, dr\wedge d\theta\wedge d\phi\;,\qquad F=\frac{Q_m}{4\pi e}\sin\theta\, d\theta\wedge d\phi\;. \tag{2.3}$$

We will seek solutions with fixed values of the charges Q_a and Q_m (we do not allow variations of these charges, for an alternative viewpoint see Ref. 7). With these assumptions the action reduces to

$$I_E=\int d\tau\left[-\frac{b\dot{b}\dot{a}}{N}-\frac{a\dot{b}^2}{2N}-\frac{Na}{2}+\frac{NQ_a^2}{2ab^2}+\frac{NaQ_m^2}{2b^2}\right]. \tag{2.4}$$

It is possible to greatly simplify this action using the change of variable

$$p=\ln a,\quad q=\ln ab,\quad \text{and } \tilde{N}=N/ab^2\,, \tag{2.5}$$

and thus obtain

$$I_e=\tfrac{1}{2}\int d\tau\left[\frac{\dot{p}^{\,2}}{\tilde{N}}-\frac{\dot{q}^{\,2}}{\tilde{N}}-\tilde{N}e^{2q}+\tilde{N}Q_a^2+\tilde{N}e^{2p}Q_m^2\right]. \tag{2.6}$$

This yields the classical equations

$$\ddot{p}-e^{2p}Q_m^2=0\,, \tag{2.7a}$$

$$\ddot{q}-e^{2q}=0 \tag{2.7b}$$

and the constraint

$$\dot{p}^{\,2}-\dot{q}^{\,2}+e^{2q}-Q_a^2-Q_m^2e^{2p}=0 \tag{2.7c}$$

in the gauge $\tilde{N}=1$. It is straightforward to solve these equations to get

$$\begin{aligned} e^{-q}&=\frac{1}{\sqrt{D}}\sinh[\sqrt{D}\,(t-t_1)],\quad D>0\\ &=\frac{1}{\sqrt{-D}}\sin[\sqrt{-D}\,(t-t_1)],\quad D<0\,, \end{aligned} \tag{2.8a}$$

$$\begin{aligned} e^{-p}&=\frac{|Q_m|}{\sqrt{C}}\sinh[\sqrt{C}\,(t-t_2)],\quad C>0\\ &=\frac{|Q_m|}{\sqrt{-C}}\sin[\sqrt{-C}\,(t-t_2)],\quad C<0\\ &=e^{\sqrt{C}t-B},\quad Q_m=0\,, \end{aligned} \tag{2.8b}$$

where C and D are constrained by the equation

$$C-D-Q_a^2=0\,; \tag{2.8c}$$

t_1, t_2, B, C, and D are constants of integration. Thus we have the following possible solutions.

(a) $Q_m=0$, $C=0$:

$$ds^2=\frac{1}{1-(Q_a/\tilde{a}\tau)^2}d\tau^2+\tilde{a}^{\,2}dr^2+\tau^2d\Omega_2^2 \tag{2.9}$$

with $\tilde{a}$ being constant and $\tau=|Q_a|/\{\tilde{a}\sin[|Q_a|(t-t_1)]\}$. This wormhole solution corresponds to a fixed radius of the S^1 and the radius of the S^2 contracts from an infinitely large value to a minimum and reexpands to infinity. The line element (2.9) covers only half the wormhole, from the asymptotically flat region to the throat although it is possible to find a metric which covers the entire manifold. The action of this configuration is $\tilde{a}b_0\pi$, $b_0=|Q_a|/\tilde{a}$ being the minimal radius of the two-sphere.

(b) $Q_a=0$, $t_1=t_2$, $C=D<0$:

$$ds^2=\frac{Q_m^2}{\tau^2+C}d\tau^2+\frac{\tau^2}{Q_m^2}dr^2+Q_m^2d\Omega_2^2 \tag{2.10}$$

and here $\tau=\sqrt{-C}/\sin[\sqrt{-C}\,(t-t_2)]$. This case corresponds to a wormhole in the S^1 direction, keeping the S^2 constant. The action of this configuration is zero.

It is also possible to find solutions where both a and b vary but no regular four geometries has the two scale factors going simultaneously to infinity.

We have also investigated the possible inclusion of a homogeneous electric field. It would have to be imaginary in Euclidean space such that its analytical continuation to Lorentzian space gives a real field. The imaginary electric field would give rise to similar equations of motion as the one from the magnetic field, so we will not discuss them further. If we would have been interested in a real Euclidean electric field we would only get a different sign for the charge squared. This latter field would reduce the effective charge $(Q_m^2-Q_e^2)^{1/2}$. No regular wormhole solution will exist for $Q_e^2>Q_m^2$ if $Q_a=0$. In the case $Q_a\neq 0$, solutions similar to the one presented in Eq. (2.9) will exist but here, a will vary. However, no solution will have both a and b going to infinity simultaneously.

III. DISCUSSION

In this Brief Report we have presented homogeneous wormhole solutions of a different topology and have calculated their action. They are solutions for either a monopole configuration of a three antisymmetric tensor or of a two antisymmetric tensor (such as electromagnetism). The solutions are asymptotically flat but not asymptotically Euclidean in the sense of Gibbons and Pope.[8]

It is interesting to realize that both of these wormholes and the one previously investigated in the literature (Refs. 1–4) are in a sense related to each other. They are all dimensional reductions of five-dimensional (5D) spaces with horizons.

The simplest example is the "Schwarzschild" black hole in five dimensions. The metric is given by

$$ds^2=-\left[1-\frac{A}{r^2}\right]dt^2+\left[1-\frac{A}{r^2}\right]^{-1}dr^2+r^2d\Omega_3^2\,, \tag{3.1}$$

where A is related to the 5D mass of the black hole. A slice of constant t gives the wormhole described in Ref. 3. The metric of the mentioned slice covers only half the wormhole but it is possible to show that it can be extended to the metric

$$ds_4^2=\left[1+\frac{b^2}{x^\mu x_\mu}\right]^2dx^\mu dx_\mu\,. \tag{3.2}$$

The metric (3.1) is the solution of the vacuum field equation in five dimensions. Using dimensional reduction we can show that the metric (3.2) is the solution of the 4D Einstein equations if a conformal field is added. Only the scalar field has a homogeneous mode on the three-sphere; therefore it is the one which has been used. It could also be an approximate solution of an electromagnetic field where the photons are in a roughly homogeneous state. Adding a cosmological constant would give a "Schwarzschild–de Sitter" type solution in 5D and its 4D analog would give the wormholes investigated in Ref. 4.

It is also possible to show that if there is a monopole on a three-sphere of a three-index tensor as a matter source in five dimensions, this will give the wormhole studied by Giddings, Strominger, and Myers.

The solutions studied in this paper are also related to space with horizons in five dimensions. The metric (2.9) comes from a 5D black holes of the form 4D *Schwarzschild* $\times R$ (such black holes were studied in Ref. 9). Finally the solution (2.10) corresponds to a 3D *anti–de Sitter* $\times S^2$ space including an electromagnetic field. Taking the appropriate section of this space gives the described wormhole.

In fact, this shows that mathematically, Misner's wormholes[10] are very similar to the one currently investigated, his are only in one less dimension.

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